$$U_{n+2} = \left[\sin t \cos^{n+1} t\right]_0^{\frac{\pi}{2}} + (n+1)\int_0^{\frac{\pi}{2}} \cos^n t \sin^2 t \, dt$$

$$U_{n+2} = (n+1)\int_0^{\frac{\pi}{2}} \cos^n t (1-\cos^2 t)\, dt \Rightarrow U_{n+2} = (n+1)(U_n - U_{n+2})$$

$$\Rightarrow U_{n+2} = \frac{n+1}{n+2} U_n .$$

3°/ Montrons par récurrence que $\forall n \in \mathbf{N}$, $U_{2n} = \frac{(2n)!}{(n!)^2} \frac{\pi}{2^{2n+1}}$.

- pour $n = 0$, $U_0 = \frac{\pi}{2} = \frac{(2\cdot 0)!}{(0!)^2} \cdot \frac{\pi}{2^{2\cdot 0+1}}$
- supposons que pour un entier donné n on a : $U_{2n} = \frac{(2n)!}{(n!)^2} \frac{\pi}{2^{2n+1}}$ et montrons que $U_{2n+2} = \frac{(2n+2)!}{((n+1)!)^2} \frac{\pi}{2^{2n+3}}$

$$U_{2n+2} = U_{(2n)+2} = \frac{2n+1}{2n+2} U_{2n} = \frac{2n+1}{2n+2} \frac{(2n)!}{(n!)^2} \frac{\pi}{2^{2n+1}}$$

$$= \frac{(2n+1)!\pi}{2^{2n+2}(n+1)(n!)^2} = \frac{(2n+2)!\pi}{2^{2n+3}((n+1)!)^2}$$

d'où $\forall n \in \mathbf{N}$, $U_{2n} = \frac{(2n)!}{(n!)^2} \frac{\pi}{2^{2n+1}}$.

4°/ a) Soit $V_n = (n+1)U_{n+1} \cdot U_n$

$$V_{n+1} = (n+2)U_{n+2} \cdot U_{n+1} = (n+2)\frac{n+1}{n+2} U_n \cdot U_{n+1}$$

$$= (n+1)U_n \cdot U_{n+1} = V_n \Rightarrow (V_n) \text{ est une suite constante.}$$

$V_0 = 1 \cdot U_1 \cdot U_0 = \frac{\pi}{2}$ d'où $\forall n \in \mathbf{N}$, $(n+1)U_n \cdot U_{n+1} = \frac{\pi}{2}$.

b) $(2n+1)U_{2n} \cdot U_{2n+1} = \frac{\pi}{2} \Rightarrow (2n+1)\frac{(2n)!}{(n!)^2} \frac{\pi}{2^{2n+1}} U_{2n+1} = \frac{\pi}{2}$

$$\Rightarrow \frac{(2n+1)!}{(n!)^2} \frac{1}{2^{2n}} U_{2n+1} = 1 \Rightarrow U_{2n+1} = \frac{(n!)^2 2^{2n}}{(2n+1)!} .$$